GROMOV K-AREA AND JUMPING CURVES IN $\mathbb{CP}^n$

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Abstract. We give here some extensions of Gromov's and Polterovich's theorems on K-area of $\mathbb{CP}^n$, particularly in the symplectic and Hamiltonian context. Our main methods involve Gromov-Witten theory, and some connections with Bott periodicity, and loop groups. The argument is closely connected with study of jumping curves in $\mathbb{CP}^n$, and as an upshot we also prove a possibly new, general theorem on these jumping curves.

1. Introduction

In [3] Gromov proposed an interesting way to probe the macroscopic geometry of Riemannian and symplectic manifolds by means of geometry of complex vector bundles on the manifold. This is a geometric analogue of the idea of K-theory which is partly why it was named K-area. This construction involves minimizing sup norm of the curvature over all homologically essential vector bundles and all connections, and his main theorem in the Riemannian setting for spin, positively curved manifolds X, g used somewhat mysteriously the index theorem for the twisted Dirac operator.

To pass to the symplectic world Gromov considered an additional variation over all compatible metrics, i.e. all compatible complex structures with a symplectic form ω on X. At the moment the resulting invariant is still very poorly understood. Here we focus on $X = \mathbb{CP}^n$ and relate this notion the classical theory of jumping curves and and quantum classes originally defined by the author in [9], and intertwined with Bott periodicity and aspects of differential geometry of loop groups, [8]. Interestingly, this allows us to arive at a purely algebraic geometric theorem in the theory of jumping curves, as well at its symplectic generalization.

Some of the symplectic methods of the present paper continue in spirit upon Polterovich [7], and Entov [2], but our viewpoint of looking at all vector bundles simultaneously is closer to Gromov's original viewpoint. Although practically this does not turn out to be very important here.

1.1. We begin by discussing K-area in the symplectic context. Fix

$$p : E \to X$$

a rank r complex vector bundle with $c_1(E) = 0$ over a closed symplectic manifold (X^{2n}, ω) and let $p : P \to X$ denote its projectivization. Let $\mathcal{A}$ be a Hamiltonian connection on P, with curvature 2-form $R^{\mathcal{A}}$, which at $x \in X$ takes values in the Lie algebra of $\mathrm{Ham}(p^{-1}(x))$, i.e. the space of normalized smooth functions on $p^{-1}(x)$. Here normalized means

$$\int_{p^{-1}(x)} G \,\mathrm{Vol}_{\omega_{st}} = 0,$$

and ω_{st} on $p^{-1}(x) \simeq \mathbb{CP}^{r-1}$ is always assumed to be the standard form with $\omega_{st}([line]) = 1$.

Let j_X be an ω-compatible almost complex structure on X, and g_{j_X} the associated metric. We define the norm of curvature by

$$||R^{\mathcal{A}}||_{g_{j_X}} = \sup_{x \in X, \xi, \eta \in T_x X} |R^{\mathcal{A}}(\xi, \eta)|^+_H, \tag{1.1}$$

where $|\cdot|^+_H$ is "half" the Hofer norm $|G|^+_H = \max G$, for $G : (p^{-1}(x) \simeq \mathbb{CP}^{r-1}) \to \mathbb{R}$ in the Lie algebra of $\mathrm{Ham}(p^{-1}(x))$, and the supremum is over all orthonormal pairs ξ, η. Here is the basic quantity we will study:

$$\text{K-area}^{-1}(X, \omega) = \inf_{E, \mathcal{A}, j_X} ||R^{\mathcal{A}}||_{g_{j_X}}, \tag{1.2}$$

where the infinum is over all E with some non-vanishing Chern number, and $2r \geq \dim_{\mathbb{R}} X$.

Remark 1.1. *The quantity* (1.2) *is closely related to the one studied by Gromov in* [3]. *This relationship is discussed in* [7]. *We make a few comments: Gromov does not projectivize and works with unitary connections and consequently with standard norm on the Lie algebra of $U(r)$, he also works with the inverse of our quantity, which we symbolize by the superscript -1 in* K-*area*$^{-1}$. *The condition that $2r \geq \dim_{\mathbb{R}} X$ is related to stability for homotopy groups of $SU(n)$ and is vacuous if we restrict to unitary connections, since after stabilizing E we may extend the unitary connection to the stabilization, without affecting the norm* (1.1).

Here is our first theorem:

Theorem 1.2.

$$\text{K-}area^{-1}(\mathbb{CP}^n, E) \equiv \inf_{\mathcal{A}, j_X} ||R^{\mathcal{A}}||_{g_{j_X}} \geq 1, \tag{1.3}$$

where the infinum is over all Hamiltonian connections $\mathcal{A}$ on projectivization of a fixed complex vector bundle E on $\mathbb{CP}^n$, provided that

$$\mathrm{rank}_{\mathbb{C}} E \geq n,$$

$c_1(E) = 0$ *and some other Chern class of E does not vanish. In particular*

$$\text{K-}area^{-1}(\mathbb{CP}^n) \geq 1.$$

This answers a question of Polterovich in [7] about finding bounds for K-area$^{-1}(\mathbb{CP}^2, E)$, for rank 2 complex vector bundle over $\mathbb{CP}^2$, with $c_1(E) = 0, c_2(E) = 2$. If c_1 does not vanish then the argument is more elementary, and already worked out in [7]. Although one can also adapt our discussion to subsume this case.

The above theorem extends:

Theorem 1.3 (Gromov, [3])**.**

$$\text{K-}area_U^{-1}(\mathbb{CP}^n, \omega_{st}) \geq 1.$$

Here U in K-area$_U^{-1}$ emphasizes that Gromov worked with unitary connections. Notice we have the same lower bound in the unitary and Hamiltonian case.

1.2. **Jumping curves in $\mathbb{CP}^n$.** Although the proof of Theorem 1.2 is via a rather transcendental piece of technology known as quantum classes, it is also closely related to the classical notion of jumping curves in $\mathbb{CP}^n$. In fact, as a corollary, we obtain an interesting phenomenon regarding these jumping curves. Here is a simplified version of the definition, suitable in our context.

Definition 1.4. *Let $E \to \mathbb{CP}^n$ be a $\mathrm{rank}_{\mathbb{C}} E = r$ holomorphic vector bundle, with $c_1(E) = 0$. A smooth rational curve C in $\mathbb{CP}^n$ will be called a* **jumping curve***, if the restriction of E to C (by which we mean pullback) is not trivial as a holomorphic vector bundle.*

Jumping curves C can be further classified by the holomorphic isomorphism type of $E|_C$, which by Grothendieck-Birkhoff theorem is:

$$E|_C \simeq \bigoplus_i \mathcal{O}(\alpha(i)), \text{ with } \sum_i \alpha(i) = 0. \tag{1.4}$$

We can actually give a symplectic generalization of this notion as follows: Let $P_E \to \mathbb{CP}^n$ denote the projectivization of E, which is a Hamiltonian bundle and so its total space has a natural deformation class of symplectic forms Ω, extending the fiber wise symplectic forms ω_{st}, see [4].

For a rational curve C in $\mathbb{CP}^n$, we have a smooth identification of $P_E|_C$ with $\mathbb{CP}^{r-1} \times S^2$, well defined in homology, since our structure group $\mathrm{Ham}(\mathbb{CP}^{r-1}, \omega_{st})$ acts trivially on homology.

Definition 1.5. *Let $E \to \mathbb{CP}^n$ be a complex vector bundle with $c_1(E) = 0$. Let J be an almost complex structure compatible with Ω on P_E, and such that the projection map $P_E \to \mathbb{CP}^n$ is J-holomorphic. We will call such J* **admissible***. A smooth rational curve C in $\mathbb{CP}^n$ is called a* **jumping curve** *if $P_E|_C \simeq \mathbb{CP}^{r-1} \times S^2$ has a J-holomorphic section in class $-d[line] + [S^2]$, $d > 0$.*

A more natural way of stating this, is that $P_E|_C$ has a J holomorphic section u, with $\langle [\widetilde{\Omega}], [u] \rangle = -d$, $d > 0$, where $[\widetilde{\Omega}]$ is the coupling class of P_E, see [4].

When J is induced by a holomorphic structure on E, this notion is equivalent to the classical notion, since in this case $P_E|_C$ is a generalized Hirzebruch bundle:

$$P_E|_C = S^3 \times_{S^1} \mathbb{CP}^{r-1},$$

for some circle subgroup $S^1 \in SU(r)$. If this subgroup is non-trivial and H denotes it's generating Hamiltonian, then points $x \in F_{\max}$, the maximum set of H, are fixed points of the S^1 action on $\mathbb{CP}^{r-1}$, and give holomorphic sections $S^3 \times_{S^1} \{x\}$ of $P_E|_C$ with above $d > 0$.

Under some conditions the locus set of classical jumping curves may be a divisor, see [6], but in general it is extremely difficult to determine this locus. We will prove the following:

Theorem 1.6. *Let $E \to \mathbb{CP}^n$ be a rank $r \geq n$, complex vector bundle, with $c_1(E) = 0$ and some other Chern class non-zero. Then for any admissible J on P_E it has a degree one jumping curves. If we also assume that J is suitably generic then d above can be chosen to be 1.*

For emphasis we give this corollary.

Corollary 1.7. *Let $E \to \mathbb{CP}^n$ be a rank $r \geq n$ holomorphic vector bundle with $c_1(E) = 0$, and some other Chern class non-zero, then it has jumping curves.*

Remark 1.8. *The condition $r \geq n$ can likely be dropped but for this would require computation of quantum classes in the "unstable" range, which at the moment is very technically difficult. However, there may of course be a less transcendental algebraic geometry approach.*

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2. Setup

Our main tool are certain characteristic cohomology classes

$$qc_k \in H^{2k}(\Omega SU(r), QH(\mathbb{CP}^{r-1})). \tag{2.1}$$

These classes were originally defined and named *quantum classes* in much more generality in [9]. Note however, that qc_k here is qc_{2k} in [9]. The grading change is for convenience, as in this context all odd quantum classes vanish. We will not need the full definition, just some basic geometric content: and the following theorem [8]:

Theorem 2.1. *The classes qc_k on $\Omega SU(r)$ are algebraically independent and generate cohomology in the stable range $2k \leq 2r-2$, with coefficients in $QH(\mathbb{CP}^{r-1})$.*

Here is a brief overview of the geometric construction of quantum classes. For more details of the following discussion see [9]. Let $M \hookrightarrow P \to X$ be a Hamiltonian bundle, with monotone fiber (M, ω) over a closed, oriented, smooth manifold X, with a Hamiltonian connection $\mathcal{A}$. (Here M is used to denote a symplectic manifold because it serves a different logical purpose to (X, ω) of Introduction, but this M will just be $\mathbb{CP}^{r-1}$ in the rest of the paper.) We have a natural S^1 action on $\Omega^2 X$, induced by rotation of S^2, along an axis of revolution containing the base point $0 \in S^2$.

Let $\Omega^2 X_{S^1}$ denote the Borel S^1 quotient:

$$\Omega^2 X_{S^1} = \Omega^2 X \times_{S^1} S^\infty. \tag{2.2}$$

Given a cycle

$$B \xrightarrow{f} \Omega^2 X_{S^1} \to \Omega^2 B\mathrm{Ham}(M, \omega)_{S^1},$$

where the second map is induced by the classifying map $X \to B\mathrm{Ham}(M, \omega)$ for the above bundle, there is a naturally induced Hamiltonian bundle $M \hookrightarrow P_f \to Y$, where $Y \to B$ is an oriented S^2-bundle over B. A quick explanation for this: any map $B \to \Omega^2 B\mathrm{Ham}(M, \omega)_{S^1}$ is induced by a map $Y \to B\mathrm{Ham}(M, \omega)$, where Y is the S^2-bundle over B, classified by the composition $B \to \Omega^2 B\mathrm{Ham}(M, \omega)_{S^1} \to \mathbb{CP}^\infty$, with the map to $\mathbb{CP}^\infty$ being the canonical projection. Equivalently we have a bundle

$$M \times S^2 \to P_f \xrightarrow{p} B,$$

with p denoting natural projection.

We may define classes

$$qc_* \in H^*(B, QH(M))$$

for P_f as in [9], via count of certain p-fiberwise holomorphic curves, with a p-fiberwise family of complex structures on P_f, induced by some Hamiltonian connection $\mathcal{A}$ on $M \hookrightarrow P \to X$. These classes are induced by universal classes

$$qc_* \in H^*(\Omega^2 B\mathrm{Ham}(M,\omega)_{S^1}, QH(M)). \tag{2.3}$$

3. Proofs

Proof of Theorem 1.6. Let E be a rank r complex vector bundle over $\mathbb{CP}^n$, with $n \leq r$, and some Chern class non-zero. We may assume without loss of generality that E has a non-vanishing Chern number. (Otherwise, restrict the following discussion to a subspace $\mathbb{CP}^i \subset \mathbb{CP}^n$, corresponding to a non-zero class $c_i(E)$.) And let

$$\mathcal{M}_{0,1}(\mathbb{CP}^n, [line], j; x_0, [\mathbb{CP}^{n-1}]) \to \mathbb{CP}^n,$$

denote the moduli space of curves with 1 free marked point and 2 fixed marked points mapping to x_0, $\mathbb{CP}^{n-1}$, $x_0 \notin \mathbb{CP}^{n-1}$, with

$$ev : \mathcal{M}_{0,1}(\mathbb{CP}^n, [line], j; x_0, [\mathbb{CP}^{n-1}]) \to \mathbb{CP}^n,$$

denoting the evaluation map given by evaluating at the free marked point. It is well known that the standard complex structure on $\mathbb{CP}^n$ is regular and that for this standard j ev is a degree one map

$$P \to \mathbb{CP}^n,$$

where P is an S^2-bundle over $\mathbb{CP}^{n-1}$ associated to the Hopf bundle. This is because there is a unique complex line through a pair of points in $\mathbb{CP}^n$. The induced cycle $f : \mathbb{CP}^{n-1} \to \Omega^2 \mathbb{CP}^n_{S^1}$ represents a class denoted a. Let $e : \mathbb{CP}^\infty \to \Omega^2 BSU(r)_{S^1} = \Omega^2 BSU(r) \times_{S^1} S^\infty$ be the section corresponding to the canonical fixed point of the S^1 action on $\Omega^2 BSU(r)$, i.e. the constant map of S^2 to the based point $x_0 \in X$.

Lemma 3.1.

$$0 \neq f_{E_*} a \in H_{2n-2}(\Omega^2 BSU(r)_{S^1}) / e_* H_*(\mathbb{CP}^\infty),$$

where

$$f_E : \Omega^2 \mathbb{CP}^n_{S^1} \to \Omega^2 BSU(r)_{S^1}$$

is the map induced by $E \to \mathbb{CP}^n$.

Proof. Let us suppose otherwise. The composition map

$$ev : P \to \mathbb{CP}^n \to BSU(r),$$

is non vanishing in homology, since E has a non-vanishing Chern number and

$$ev : P \to \mathbb{CP}^n$$

is degree one by discussion above.

Let

$$H : T \to \Omega^2 BSU(r)_{S^1}$$

be a bordism of $f_{E_*} a_{i-1}$ to $c \in e_* H_*(\mathbb{CP}^\infty)$. We'll call the corresponding boundary pieces of T by T_a and T_c. Consequently, the bordism H induces an S^2 bundle P_T over T, it is the pull-back of the tautological S^2 bundle

$$(\Omega^2_{x_0} BSU(r) \times E^\infty) \times_{S^1} S^2 \to \Omega^2_{x_0} BSU(r)_{S^1}.$$

and of course P_T restricts over T_a to P. We have a natural "evaluation" map

$$ev_T : P_T \to BSU(r), \tag{3.1}$$

restricting to evaluation maps ev, ev_c over boundary, and so a homology of $[ev]$ to $[ev_c]$, but ev_c is the constant map to the based point $x_0 \in BU$ a contradiction. $\square$

Lemma 3.2. *For some α_i, β_i*

$$\langle \prod_{i,j} qc^{\alpha_i}_{\beta_i} \wedge l^j, f_{E_*} a_{n-1} \rangle \neq 0. \tag{3.2}$$

Proof. Note that all of cohomology of $\Omega^2 BSU(r) \simeq \Omega SU(r)$, is in even degree, since by Milnor-Morre, Cartan-Serre [5], [1], the rational homology algebra is generated as a ring with Pontryagin product by the rational homotopy groups, (via Hurewicz homomorphism) which are all in even degrees since the rational homotopy groups of $SU(r)$ are well known to be all in odd degrees. (In fact $SU(r)$ has the rational homotopy type of the product of odd spheres $S^3 \times S^5 \ldots$) Consequently, the Serre spectral sequence degenerates at the second page and so:

$$H^*(\Omega^2 BSU(r)_{S^1}) \simeq H^*(\Omega^2 BSU(r)) \otimes H^*(\mathbb{CP}^\infty) \simeq H^*(\Omega SU(r)) \otimes H^*(\mathbb{CP}^\infty). \tag{3.3}$$

Our lemma then follows by Lemma 3.1 and Theorem 2.1. $\square$

The theorem then readily follows. Since by construction of quantum classes and Lemma 3.2, for any fixed (not necessarily regular) complex structure J on P compatible with Ω, and with projection to $\mathbb{CP}^n$ for some complex line l in $\mathbb{CP}^n$, the restriction of P to l, which is diffeomorphic to $\mathbb{CP}^{r-1} \times S^2$ has a J holomorphic stable section u in total class $S = -[line] + S^2$. As otherwise the relavant Gromov-Witten invariants in class S all vanish and (3.2) is impossible. Of course the stable section u may be in the form of a holomorphic section u_p in class $-d[line] + S^2$, with $d > 1$ together with some vertical holomorphic bubbles, but this still implies our claim. $\square$

Proof of Theorem 1.2. We just need the following lemma:

Lemma 3.3. *The norm of the curvature $||R_{\mathcal{A}}||$ of the projectivization $\mathbb{CP}^{r-1} \hookrightarrow P \to \mathbb{CP}^n$ is at least* 1.

Proof. Let $\widetilde{\Omega}$ denote the coupling form of the Hamiltonian fibration P associated to $\mathcal{A}$, (see for example [4] for discussion on coupling forms). This is a certain closed form associated to the curvature form of $\mathcal{A}$, with the following properties.

The restriction of $\widetilde{\Omega}$ to fibers $M \simeq \mathbb{CP}^{r-1}$ of $P \to \mathbb{CP}^n$ coincides with ω_{st}. The $\widetilde{\Omega}$-orthogonal subspaces in TP to the fibers are horizontal subspaces, whose value on horizontal lifts

$$\widetilde{v}, \widetilde{w} \in T_{m,z} P$$

of $v, w \in T_z \mathbb{CP}^n$ are given by

$$\widetilde{\Omega}(\widetilde{v}, \widetilde{w}) = -R_{\mathcal{A}}(v, w)(m),$$

for $m \in M_z$; in other words we evaluate the Lie algebra element of $\mathrm{Ham}(M_z, \omega)$: $R_{\mathcal{A}}(v, w)$ (i.e. a function on M_z) at m.

Consider the symplectic form $\Omega = \widetilde{\Omega} + (||R^{\mathcal{A}}|| + \epsilon)\omega^n_{st}$, where ω^n_{st}, the standard Fubini-Study symplectic form on the base normalized by the condition that the area of a complex line is 1, and $\epsilon > 0$. Pick any compatible complex structure $J_{\mathcal{A}}$. By Theorem 1.6, for some complex line l in $\mathbb{CP}^n$, the restriction of P to l, which is diffeomorphic to $\mathbb{CP}^{r-1} \times S^2$ has a $J_{\mathcal{A}}$ holomorphic section u in class

$S = -d \cdot [line] + S^2$, with $d \geq 1$. Where $[line]$ is the class of the complex line in $\mathbb{CP}^{r-1}$.

Since $J_{\mathcal{A}}$ is Ω compatible, for the class S, $J_{\mathcal{A}}$-holomorphic section u of $P|_l$ we get

$$0 \leq [\Omega]([u]) = [\widetilde{\Omega}]([u]) + ||R^{\mathcal{A}}|| + \epsilon. \tag{3.4}$$

On the other hand $[\widetilde{\Omega}] = [\omega_{st}]$ on $P|_l$ since the cohomology class of the coupling form is independent of the choice of connection, and the form ω_{st} on $\mathbb{CP}^{r-1} \times S^2$, is another coupling form. Since $[\omega_{st}]([line]) = 1$ by our normalization, it follows that $[\widetilde{\Omega}]([u]) = -d$, since $[u] = -[line] + S^2$. So we get:

$$d \leq ||R^{\mathcal{A}}|| + \epsilon, \tag{3.5}$$

for every $\epsilon > 0$. □

This finishes the proof of the theorem. □

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